Mathematics

(Chapter – 13) (Limits and Derivatives)

(Class – XI)

Exercise 13.1

Question 1:

Evaluate the Given $\lim_{x \to 3} x+3$ limit:

Answer 1:

$\lim_{x \to 3} x+3 = 3+3 = 6$

Question 2:

Evaluate the Given limit: $\lim_{x \to \pi}\left(x - \frac{22}{7}\right)$

Answer 2:

$\lim_{x \to \pi}\left(x - \frac{22}{7}\right) = \left(\pi - \frac{22}{7}\right)$

Question 3:

Evaluate the Given limit: $\lim_{r \to 1} \pi r^2$

Answer 3:

$\lim_{r \to 1} \pi r^2 = \pi (1)^2 = \pi$

Question 4:

Evaluate the Given limit: $\lim_{x \to 4} \frac{4x+3}{x-2}$

Answer 4:

$\lim_{x \to 4} \frac{4x+3}{x-2} = \frac{4(4)+3}{4-2} = \frac{16+3}{2} = \frac{19}{2}$

Question 5:

Evaluate the Given limit: $\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$

Answer 5:

$\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1} = \frac{(-1)^{10}+(-1)^5+1}{-1-1} = \frac{1-1+1}{-2} = -\frac{1}{2}$

Question 6:
Evaluate the Given limit: $\lim_{x\to 0}\frac{(x+1)^5-1}{x}$

Answer 6:

$$\lim_{x\to 0}\frac{(x+1)^5-1}{x}$$

Put $x + 1 = y$ so that $y \to 1$ as $x \to 0$.

$$\text{Accordingly, } \lim_{x\to 0}\frac{(x+1)^5-1}{x} = \lim_{y\to 1}\frac{y^5-1}{y-1}$$

$$= \lim_{y\to 1}\frac{y^5-1^5}{y-1}$$

$$= 5.1^{5-1} \qquad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 5$$

$$\therefore \lim_{x\to 0}\frac{(x+5)^5-1}{x} = 5$$

Question 7:
Evaluate the Given limit: $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

Answer 7:

At $x = 2$, the value of the given rational function takes the form. $\frac{0}{0}$

$$\therefore \lim_{x\to 2}\frac{3x^2-x-10}{x^2-4} = \lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

$$= \lim_{x\to 2}\frac{3x+5}{x+2}$$

$$= \frac{3(2)+5}{2+2}$$

$$= \frac{11}{4}$$

Question 8:

Evaluate the Given limit: $\lim\limits_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Answer 8:

At x = 2, the value of the given rational function takes the form. $\frac{0}{0}$

$$\therefore \lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$$
$$=\lim_{x\to 3}\frac{(x+3)(x^2+9)}{2x+1}$$
$$=\frac{(3+3)(3^2+9)}{2(3)+1}$$
$$=\frac{6\times 18}{7}$$
$$=\frac{108}{7}$$

Question 9:

Evaluate the Given limit: $\lim\limits_{x\to 0}\frac{ax+b}{cx+1}$

Answer 9:

$$\lim_{x\to 0}\frac{ax+b}{cx+1}=\frac{a(0)+b}{c(0)+1}=b$$

Question 10:

Evaluate the Given limit: $\lim\limits_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Answer 10:

$$\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$$

At z = 1, the value of the given function takes the form. $\frac{0}{0}$

Put $z^{\frac{1}{6}}=x$ so that $z\to 1$ as $x\to 1$.

Accordingly, $\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

$= \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1}$

$= 2.1^{2-1}$ $\left[\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$

$= 2$

$\therefore \lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = 2$

Question 11:
Evaluate the Given limit: $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$

Answer 11:

$$\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a} = \frac{a(1)^2 + b(1) + c}{c(1)^2 + b(1) + a}$$

$$= \frac{a + b + c}{a + b + c}$$

$$= 1 \qquad [a + b + c \neq 0]$$

Question 12:
Evaluate the Given limit: $\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

Answer 12:

$$\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$$

At $x = -2$, the value of the given function takes the form. $\frac{0}{0}$

$$\text{Now, } \lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}=\lim_{x\to -2}\frac{\left(\frac{2+x}{2x}\right)}{x+2}$$

$$=\lim_{x\to -2}\frac{1}{2x}$$

$$=\frac{1}{2(-2)}=\frac{-1}{4}$$

Question 13:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{bx}$

Answer 13:

$$\lim_{x\to 0}\frac{\sin ax}{bx}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$

$$\text{Now, } \lim_{x\to 0}\frac{\sin ax}{bx}=\lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{ax}{bx}$$

$$=\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)\times\left(\frac{a}{b}\right)$$

$$=\frac{a}{b}\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right) \qquad [x\to 0\Rightarrow ax\to 0]$$

$$=\frac{a}{b}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=\frac{a}{b}$$

Question 14:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ a,\ b\neq 0$

Answer 14:

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ a,\ b\neq 0$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$

$$\text{Now, } \lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\times ax}{\left(\frac{\sin bx}{bx}\right)\times bx}$$

$$=\left(\frac{a}{b}\right)\times\frac{\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim_{bx\to 0}\left(\frac{\sin bx}{bx}\right)} \qquad \left[\begin{array}{l}x\to 0\Rightarrow ax\to 0\\ \text{and } x\to 0\Rightarrow bx\to 0\end{array}\right]$$

$$=\left(\frac{a}{b}\right)\times\frac{1}{1} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=\frac{a}{b}$$

Question 15:

Evaluate the Given limit: $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Answer 16:

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

It is seen that $x\to\pi\Rightarrow(\pi-x)\to 0$

$$\therefore \lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\frac{1}{\pi}\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{(\pi-x)}$$

$$=\frac{1}{\pi}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=\frac{1}{\pi}$$

Question 16:

Evaluate the given limit: $\lim_{x\to 0}\frac{\cos x}{\pi - x}$

Answer 16:

$$\lim_{x\to 0}\frac{\cos x}{\pi-x}=\frac{\cos 0}{\pi-0}=\frac{1}{\pi}$$

Question 17:

Evaluate the Given limit: $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

Answer 17:

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$$

At x = 0, the value of the given function takes the form. $\frac{0}{0}$

Now,

$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}=\lim_{x\to 0}\frac{1-2\sin^2 x-1}{1-2\sin^2\frac{x}{2}-1} \qquad \left[\cos x=1-2\sin^2\frac{x}{2}\right]$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\left(\frac{\sin^2 x}{x^2}\right)\times x^2}{\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)\times\frac{x^2}{4}}$$

$$=4\frac{\lim_{x\to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$=4\frac{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2}{\left(\lim_{\frac{x}{2}\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2} \qquad \left[x\to 0\Rightarrow\frac{x}{2}\to 0\right]$$

$$=4\frac{1^2}{1^2} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=4$$

Question 18:

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$

Answer 18:

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$$

At $x = 0$, the value of the given function takes the form. $\frac{0}{0}$

Now,

$$\begin{aligned}
\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x} &= \frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}\\
&= \frac{1}{b}\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}(a+\cos x)\\
&= \frac{1}{b}\times\frac{1}{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)}\times\lim_{x\to 0}(a+\cos x)\\
&= \frac{1}{b}\times(a+\cos 0) \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]\\
&= \frac{a+1}{b}
\end{aligned}$$

Question 19:

Evaluate the Given limit: $\lim_{x\to 0} x\sec x$

Answer 19:

$$\lim_{x\to 0} x\sec x = \lim_{x\to 0}\frac{x}{\cos x}=\frac{0}{\cos 0}=\frac{0}{1}=0$$

Question 20:

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$ $a, b, a+b \neq 0$

Answer 20:

At x = 0, the value of the given function takes the form. $\frac{0}{0}$

Now,

$$\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin ax}{ax}\right) ax + bx}{ax + bx\left(\frac{\sin bx}{bx}\right)}$$

$$= \frac{\left(\lim_{ax \to 0} \frac{\sin ax}{ax}\right) \times \lim_{x \to 0}(ax) + \lim_{x \to 0} bx}{\lim_{x \to 0} ax + \lim_{x \to 0} bx \left(\lim_{bx \to 0} \frac{\sin bx}{bx}\right)} \qquad [\text{As } x \to 0 \Rightarrow ax \to 0 \text{ and } bx \to 0]$$

$$= \frac{\lim_{x \to 0}(ax) + \lim_{x \to 0} bx}{\lim_{x \to 0} ax + \lim_{x \to 0} bx} \qquad \left[\lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \frac{\lim_{x \to 0}(ax + bx)}{\lim_{x \to 0}(ax + bx)}$$

$$= \lim_{x \to 0}(1)$$

$$= 1$$

Question 21:

Evaluate the Given limit: $\lim_{x\to 0}\left(\operatorname{cosec} x - \cot x\right)$

Answer 21:

At $x = 0$, the value of the given function takes the form. $\infty - \infty$

Now,

$$\lim_{x\to 0}\left(\operatorname{cosec} x - \cot x\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$

$$= \lim_{x\to 0}\frac{\left(\frac{1-\cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$

$$= \frac{\lim_{x\to 0}\frac{1-\cos x}{x}}{\lim_{x\to 0}\frac{\sin x}{x}}$$

$$= \frac{0}{1} \qquad \left[\lim_{x\to 0}\frac{1-\cos x}{x} = 0 \text{ and } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= 0$$

Question 22:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

Answer 22:

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

At $x=\frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$.

Now, put $x-\frac{\pi}{2}=y$ so that $x\to\frac{\pi}{2},\ y\to 0$.

$$\therefore \lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}=\lim_{y\to 0}\frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$=\lim_{y\to 0}\frac{\tan(\pi+2y)}{y}$$

$$=\lim_{y\to 0}\frac{\tan 2y}{y} \qquad [\tan(\pi+2y)=\tan 2y]$$

$$=\lim_{y\to 0}\frac{\sin 2y}{y\cos 2y}$$

$$=\lim_{y\to 0}\left(\frac{\sin 2y}{2y}\times\frac{2}{\cos 2y}\right)$$

$$=\left(\lim_{2y\to 0}\frac{\sin 2y}{2y}\right)\times\lim_{y\to 0}\left(\frac{2}{\cos 2y}\right) \qquad [y\to 0\Rightarrow 2y\to 0]$$

$$=1\times\frac{2}{\cos 0} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=1\times\frac{2}{1}$$

$$=2$$

Question 23:

Find $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

Answer 23:

The given function is $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0}[2x+3] = 2(0)+3 = 3$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} 3(x+1) = 3(0+1) = 3$$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} f(x) = 3$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} 3(x+1) = 3(1+1) = 6$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} 3(x+1) = 3(1+1) = 6$$

$$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = 6$$

Question 24:

Find $\lim\limits_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x^2-1, & x > 1 \end{cases}$

Answer 24:

The given function is

$$f(x)=\begin{cases} x^2-1, & x\le 1 \\ -x^2-1, & x>1 \end{cases}$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}\left[x^2-1\right]=1^2-1=1-1=0$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}\left[-x^2-1\right]=-1^2-1=-1-1=-2$$

It is observed that $\lim_{x\to 1^-} f(x)\ne \lim_{x\to 1^+} f(x)$.

Hence, $\lim_{x\to 1} f(x)$ does not exist.

Question 25:

Evaluate $\lim_{x\to 0}$ $f(x)$, where $f(x)$ = $\begin{cases} \frac{|x|}{x}, & x\ne 0 \\ 0, & x=0 \end{cases}$

Answer 25:

The given function is $f(x)$ = $\begin{cases} \frac{|x|}{x}, & x\ne 0 \\ 0, & x=0 \end{cases}$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left[\frac{|x|}{x}\right] \\ &= \lim_{x\to 0}\left(\frac{-x}{x}\right) \qquad \left[\text{When } x \text{ is negaitve, } |x|=-x\right] \\ &= \lim_{x\to 0}(-1) \\ &= -1\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\left[\frac{|x|}{x}\right] \\ &= \lim_{x\to 0}\left[\frac{x}{x}\right] \qquad \left[\text{When } x \text{ is positive, } |x|=x\right] \\ &= \lim_{x\to 0}(1) \\ &= 1\end{aligned}$$

It is observed that $\lim_{x\to 0^-} f(x)\ne \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question 26:

Find $\lim_{x \to 0} f(x)$, where $f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Answer 26:

The given function is

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \left[\frac{x}{|x|} \right] \\ &= \lim_{x \to 0} \left[\frac{x}{-x} \right] && \left[\text{When } x < 0,\ |x| = -x \right] \\ &= \lim_{x \to 0} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left[\frac{x}{|x|} \right] \\ &= \lim_{x \to 0} \left[\frac{x}{x} \right] && \left[\text{When } x > 0,\ |x| = x \right] \\ &= \lim_{x \to 0} (1) \\ &= 1 \end{aligned}$$

It is observed that $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$.

Hence, $\lim_{x \to 0} f(x)$ does not exist.

Question 27:

Find $\lim_{x \to 5} f(x)$, where $f(x) = |x| - 5$

Answer 27:

The given function is $f(x) = |x| - 5$.

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-}\left[|x| - 5\right]$$

$$= \lim_{x \to 5}(x-5) \qquad \left[\text{When } x > 0,\ |x| = x\right]$$

$$= 5 - 5$$

$$= 0$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+}\left(|x| - 5\right)$$

$$= \lim_{x \to 5}(x-5) \qquad \left[\text{When } x > 0,\ |x| = x\right]$$

$$= 5 - 5$$

$$= 0$$

$$\therefore \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = 0$$

$$\text{Hence, } \lim_{x \to 5} f(x) = 0$$

Question 28:

Suppose $f(x) = \begin{cases} a+bx, & if x < 1 \\ 4, & if\ x = 0 \\ b-ax, & if x > 1 \end{cases}$,

and $\lim_{x\to 1} f(x) = f(1)$ what are possible values of *a* and *b*?

Answer 28:

The given function is

$$f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(a+bx) = a+b$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(b-ax) = b-a$

$f(1) = 4$

It is given that $\lim_{x\to 1} f(x) = f(1)$.

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1)$

$\Rightarrow a+b=4$ and $b-a=4$

On solving these two equations, we obtain $a=0$ and $b=4$.

Thus, the respective possible values of *a* and *b* are 0 and 4.

Question 29:

Let a_1, a_2, . . ., a_n be fixed real numbers and define a function

$f(x) = (x - a_1)(x - a_2)...(x - a_n)$.

What is $\lim_{x \to a_1} f(x)$? For some $a \neq a_1, a_2... a_n$, compute $\lim_{x \to a} f(x)$.

Answer 29:

The given function is $f(x) = (x - a_1)(x - a_2)...(x - a_n)$.

$$\lim_{x \to a_1} f(x) = \lim_{x \to a_1}\left[(x - a_1)(x - a_2)...(x - a_n)\right]$$

$$= \left[\lim_{x \to a_1}(x - a_1)\right]\left[\lim_{x \to a_1}(x - a_2)\right]...\left[\lim_{x \to a_1}(x - a_n)\right]$$

$$= (a_1 - a_1)(a_1 - a_2)...(a_1 - a_n) = 0$$

$$\therefore \lim_{x \to a_1} f(x) = 0$$

$$\text{Now, } \lim_{x \to a} f(x) = \lim_{x \to a}\left[(x - a_1)(x - a_2)...(x - a_n)\right]$$

$$= \left[\lim_{x \to a}(x - a_1)\right]\left[\lim_{x \to a}(x - a_2)\right]...\left[\lim_{x \to a}(x - a_n)\right]$$

$$= (a - a_1)(a - a_2).....(a - a_n)$$

$$\therefore \lim_{x \to a} f(x) = (a - a_1)(a - a_2)...(a - a_n)$$

Question 30:

If $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$.

For what value(s) of a does $\lim_{x \to a} f(x)$ exists?

Answer 30:

The given function is

$$f(x)=\begin{cases}|x|+1, & x<0\\ 0, & x=0\\ |x|-1, & x>0\end{cases}$$

When $a=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(|x|+1)$$
$$=\lim_{x\to 0}(-x+1) \qquad [\text{If } x<0,\ |x|=-x]$$
$$=-0+1$$
$$=1$$
$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(|x|-1)$$
$$=\lim_{x\to 0}(x-1) \qquad [\text{If } x>0,\ |x|=x]$$
$$=0-1$$
$$=-1$$

Here, it is observed that $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

When a < 0,

$$\lim_{x\to a^-} f(x)=\lim_{x\to a^-}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [x<a<0 \Rightarrow |x|=-x]$$
$$=-a+1$$
$$\lim_{x\to a^+} f(x)=\lim_{x\to a^+}(|x|+1)$$
$$=\lim_{x\to a}(-x+1) \qquad [a<x<0 \Rightarrow |x|=-x]$$
$$=-a+1$$
$$\therefore \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=-a+1$$

Thus, limit of $f(x)$ exists at $x=a$, where $a<0$.

When a > 0

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-}(|x|-1)$$
$$= \lim_{x\to a}(x-1) \qquad [0 < x < a \Rightarrow |x| = x]$$
$$= a-1$$
$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+}(|x|-1)$$
$$= \lim_{x\to a}(x-1) \qquad [0 < a < x \Rightarrow |x| = x]$$
$$= a-1$$
$$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = a-1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$.

Thus, $\lim_{x\to a} f(x)$ exists for all $a \neq 0$.

Question 31:

If the function *f*(*x*) satisfies, $\lim_{x\to 1}\frac{f(x)-2}{x^2-1} = \pi$ evaluate $\lim_{x\to 1} f(x)$.

Answer 31:

$$\lim_{x\to 1}\frac{f(x)-2}{x^2-1} = \pi$$
$$\Rightarrow \frac{\lim_{x\to 1}(f(x)-2)}{\lim_{x\to 1}(x^2-1)} = \pi$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = \pi \lim_{x\to 1}(x^2-1)$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = \pi(1^2-1)$$
$$\Rightarrow \lim_{x\to 1}(f(x)-2) = 0$$
$$\Rightarrow \lim_{x\to 1} f(x) - \lim_{x\to 1} 2 = 0$$
$$\Rightarrow \lim_{x\to 1} f(x) - 2 = 0$$
$$\therefore \lim_{x\to 1} f(x) = 2$$

Question 32:

If. $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$ For what integers *m* and *n* does

$\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist?

Answer 32: The given function is

$$f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0}\left(mx^2+n\right) \\ &= m(0)^2+n \\ &= n\end{aligned}$$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0}\left(nx+m\right) \\ &= n(0)+m \\ &= m.\end{aligned}$$

Thus, $\lim_{x\to 0} f(x)$ exists if $m = n$.

$$\begin{aligned}\lim_{x\to 1^-} f(x) &= \lim_{x\to 1}\left(nx+m\right) \\ &= n(1)+m \\ &= m+n\end{aligned}$$

$$\begin{aligned}\lim_{x\to 1^+} f(x) &= \lim_{x\to 1}\left(nx^3+m\right) \\ &= n(1)^3+m \\ &= m+n\end{aligned}$$

$$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x).$$

Thus $\lim_{x\to 1} f(x)$ exist for any integral value of m and n.

Exercise 13.2

Question 1:
Find the derivative of $x^2 - 2$ at $x = 10$.
Answer 1:
Let $f(x) = x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$
$$= \lim_{h\to 0}\frac{\left[(10+h)^2-2\right]-(10^2-2)}{h}$$
$$= \lim_{h\to 0}\frac{10^2+2.10.h+h^2-2-10^2+2}{h}$$
$$= \lim_{h\to 0}\frac{20h+h^2}{h}$$
$$= \lim_{h\to 0}(20+h) = (20+0) = 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2:
Find the derivative of $99x$ at $x = 100$.
Answer 2:
Let $f(x) = 99x$. Accordingly,

$$f'(100) = \lim_{h\to 0}\frac{f(100+h)-f(100)}{h}$$
$$= \lim_{h\to 0}\frac{99(100+h)-99(100)}{h}$$
$$= \lim_{h\to 0}\frac{99\times 100+99h-99\times 100}{h}$$
$$= \lim_{h\to 0}\frac{99h}{h}$$
$$= \lim_{h\to 0}(99) = 99$$

Thus, the derivative of $99x$ at $x = 100$ is 99.

Question 3:
Find the derivative of x at $x = 1$.
Answer 3:
Let $f(x) = x$. Accordingly,

$$f'(1) = \lim_{h\to0}\frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to0}\frac{(1+h)-1}{h}$$
$$= \lim_{h\to0}\frac{h}{h}$$
$$= \lim_{h\to0}(1)$$
$$= 1$$

Thus, the derivative of x at $x = 1$ is 1.

Question 4:
Find the derivative of the following functions from first principle.
(i) $x^3 - 27$
(ii) $(x - 1)(x - 2)$
(iii) $\frac{1}{x^2}$
(iv) $\frac{x+1}{x-1}$

Answer 4:
(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to0}\frac{\left[(x+h)^3-27\right]-\left(x^3-27\right)}{h}$$
$$= \lim_{h\to0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h}$$
$$= \lim_{h\to0}\frac{h^3+3x^2h+3xh^2}{h}$$
$$= \lim_{h\to0}\left(h^2+3x^2+3xh\right)$$
$$= 0+3x^2+0 = 3x^2$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
&= \lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h}\\
&= \lim_{h\to 0}\frac{\left(x^2+hx-2x+hx+h^2-2h-x-h+2\right)-\left(x^2-2x-x+2\right)}{h}\\
&= \lim_{h\to 0}\frac{\left(hx+hx+h^2-2h-h\right)}{h}\\
&= \lim_{h\to 0}\frac{2hx+h^2-3h}{h}\\
&= \lim_{h\to 0}(2x+h-3)\\
&= (2x+0-3)\\
&= 2x-3
\end{aligned}
$$

(iii) Let $f(x)=\frac{1}{x^2}$

Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-h^2-2hx}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\left[\frac{-h-2x}{x^2(x+h)^2}\right]$$

$$=\frac{0-2x}{x^2(x+0)^2}=\frac{-2}{x^3}$$

(iv) Let $f(x)=\frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$

$$= \lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$

$$= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}$$

Question 5:

For the function

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$$

Prove that $f'(1)=100f'(0)$

Answer 5:

The given function is

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1\right]$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+\ldots+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$, we obtain

$$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+\ldots+\frac{2x}{2}+1+0$$

$$=x^{99}+x^{98}+\ldots+x+1$$

$$\therefore f'(x)=x^{99}+x^{98}+\ldots+x+1$$

At $x=0$,

$$f'(0)=1$$

At $x=1$,

$$f'(1)=1^{99}+1^{98}+\ldots+1+1=\left[1+1+\ldots+1+1\right]_{100\text{ terms}}=1\times100=100$$

Thus, $f'(1)=100\times f^{1}(0)$

Question 6:

Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+\ldots+a^{n-1}x+a^n$ for some fixed real number *a*.

Answer 6:

Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$

$$\therefore f'(x)=\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n\right)$$
$$=\frac{d}{dx}\left(x^n\right)+a\frac{d}{dx}\left(x^{n-1}\right)+a^2\frac{d}{dx}\left(x^{n-2}\right)+...+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}+a^n(0)$$
$$=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}$$

Question 7:

For some constants *a* and *b*, find the derivative of

(i) $(x-a)(x-b)$ (ii) $(ax^2+b)^2$ (iii) $\frac{x-a}{x-b}$

Answer 7:

(i) Let $f(x)=(x-a)(x-b)$

$$\Rightarrow f(x)=x^2-(a+b)x+ab$$
$$\therefore f'(x)=\frac{d}{dx}\left(x^2-(a+b)x+ab\right)$$
$$=\frac{d}{dx}\left(x^2\right)-(a+b)\frac{d}{dx}(x)+\frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$, we obtain

$$f'(x)=2x-(a+b)+0=2x-a-b$$

(ii) Let $f(x)=\left(ax^2+b\right)^2$

$$\Rightarrow f(x)=a^2x^4+2abx^2+b^2$$

$$\therefore f'(x)=\frac{d}{dx}\left(a^2x^4+2abx^2+b^2\right)=a^2\frac{d}{dx}\left(x^4\right)+2ab\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(b^2\right)$$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$$f'(x)=a^2\left(4x^3\right)+2ab(2x)+b^2(0)$$

$$=4a^2x^3+4abx$$

$$=4ax\left(ax^2+b\right)$$

(iii) Let $f(x)=\frac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x)=\frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x)=\frac{(x-b)\frac{d}{dx}(x-a)-(x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$=\frac{(x-b)(1)-(x-a)(1)}{(x-b)^2}$$

$$=\frac{x-b-x+a}{(x-b)^2}$$

$$=\frac{a-b}{(x-b)^2}$$

Question 8:

Find the derivative of $\frac{x^n - a^n}{x - a}$ for some constant *a*.

Answer 8:

Let $f(x) = \frac{x^n - a^n}{x - a}$

$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x^n - a^n}{x - a}\right)$

By quotient rule,

$$f'(x) = \frac{(x-a)\frac{d}{dx}(x^n - a^n) - (x^n - a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

Question 9:

Find the derivative of

(i) $2x - \frac{3}{4}$

(ii) $(5x^3 + 3x - 1)(x - 1)$

(iii) $x^{-3}(5 + 3x)$

(iv) $x^5(3 - 6x^{-9})$

(v) $x^{-4}(3 - 4x^{-5})$

(vi) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Answer 9:

(i) Let $f(x)=2x-\frac{3}{4}$

$$\begin{aligned}f'(x)&=\frac{d}{dx}\left(2x-\frac{3}{4}\right)\\&=2\frac{d}{dx}(x)-\frac{d}{dx}\left(\frac{3}{4}\right)\\&=2-0\\&=2\end{aligned}$$

(ii) Let $f(x) = (5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$\begin{aligned}f'(x)&=\left(5x^3+3x-1\right)\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}\left(5x^3+3x-1\right)\\&=\left(5x^3+3x-1\right)(1)+(x-1)\left(5.3x^2+3-0\right)\\&=\left(5x^3+3x-1\right)+(x-1)\left(15x^2+3\right)\\&=5x^3+3x-1+15x^3+3x-15x^2-3\\&=20x^3-15x^2+6x-4\end{aligned}$$

(iii) Let $f(x) = x^{-3}(5 + 3x)$

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= x^{-3}\frac{d}{dx}(5+3x)+(5+3x)\frac{d}{dx}(x^{-3})\\
&= x^{-3}(0+3)+(5+3x)(-3x^{-3-1})\\
&= x^{-3}(3)+(5+3x)(-3x^{-4})\\
&= 3x^{-3}-15x^{-4}-9x^{-3}\\
&= -6x^{-3}-15x^{-4}\\
&= -3x^{-3}\left(2+\frac{5}{x}\right)\\
&= \frac{-3x^{-3}}{x}(2x+5)\\
&= \frac{-3}{x^{4}}(5+2x)
\end{aligned}
$$

(iv) Let $f(x) = x^5(3 - 6x^{-9})$

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= x^{5}\frac{d}{dx}(3-6x^{-9})+(3-6x^{-9})\frac{d}{dx}(x^{5})\\
&= x^{5}\left\{0-6(-9)x^{-9-1}\right\}+(3-6x^{-9})(5x^{4})\\
&= x^{5}(54x^{-10})+15x^{4}-30x^{-5}\\
&= 54x^{-5}+15x^{4}-30x^{-5}\\
&= 24x^{-5}+15x^{4}\\
&= 15x^{4}+\frac{24}{x^{5}}
\end{aligned}
$$

(v) Let $f(x) = x^{-4}(3 - 4x^{-5})$

By Leibnitz product rule,

$$f'(x) = x^{-4}\frac{d}{dx}(3-4x^{-5}) + (3-4x^{-5})\frac{d}{dx}(x^{-4})$$

$$= x^{-4}\{0-4(-5)x^{-5-1}\} + (3-4x^{-5})(-4)x^{-4-1}$$

$$= x^{-4}(20x^{-6}) + (3-4x^{-5})(-4x^{-5})$$

$$= 20x^{-10} - 12x^{-5} + 16x^{-10}$$

$$= 36x^{-10} - 12x^{-5}$$

$$= -\frac{12}{x^5} + \frac{36}{x^{10}}$$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right) - \frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x+1)}{(x+1)^2}\right] - \left[\frac{(3x-1)\frac{d}{dx}(x^2) - x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$

$$= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2}\right] - \left[\frac{(3x-1)(2x) - (x^2)(3)}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x^2}{(3x-1)^2}\right]$$

$$= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

Question 10:

Find the derivative of cos *x* from first principle.

Answer 10:

Let *f* (*x*) = cos *x*. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=\lim_{h\to 0}\left[\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$

$$=-\cos x\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$=-\cos x(0)-\sin x(1)\qquad\left[\lim_{h\to 0}\frac{1-\cos h}{h}=0\text{ and }\lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$=-\sin x$$

$$\therefore f'(x)=-\sin x$$

Question 11:

Find the derivative of the following functions:

(i) sin *x* cos *x* (ii) sec *x* (iii) 5 sec *x* + 4 cos *x*

(iv) cosec *x* (v) 3cot *x* + 5cosec *x* (vi) 5sin *x* − 6cos *x* + 7

(vii) 2tan *x* − 7sec *x*

Answer 11:

(i) Let $f(x) = \sin x \cos x$.

Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h)\cos(x+h)-\sin x\cos x}{h} \\
&= \lim_{h\to 0}\frac{1}{2h}\left[2\sin(x+h)\cos(x+h)-2\sin x\cos x\right] \\
&= \lim_{h\to 0}\frac{1}{2h}\left[\sin 2(x+h)-\sin 2x\right] \\
&= \lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2}\cdot\sin\frac{2x+2h-2x}{2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\cos(2x+h)\sin h\right] \\
&= \lim_{h\to 0}\cos(2x+h).\lim_{h\to 0}\frac{\sin h}{h} \\
&= \cos(2x+0).1 \\
&= \cos 2x
\end{aligned}
$$

(ii) Let $f(x) = \sec x$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)} \\
&= \frac{1}{\cos x}.\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \\
&= \frac{1}{\cos x}.1.\frac{\sin x}{\cos x} \\
&= \sec x\tan x
\end{aligned}
$$

(iii) Let *f* (*x*) = 5 sec *x* + 4 cos *x*. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{5\sec(x+h)+4\cos(x+h)-[5\sec x+4\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{[\sec(x+h)-\sec x]}{h}+4\lim_{h\to 0}\frac{[\cos(x+h)-\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]+4\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos x\cos h-\sin x\sin h-\cos x]$$

$$=\frac{5}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}\left[-\cos x(1-\cos h)-\sin x\sin h\right]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]+4\left[-\cos x\lim_{h\to 0}\frac{(1-\cos h)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$=\frac{5}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)}\right]+4\left[(-\cos x).(0)-(\sin x).1\right]$$

$$=\frac{5}{\cos x}.\left[\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]-4\sin x$$

$$=\frac{5}{\cos x}.\frac{\sin x}{\cos x}.1-4\sin x$$

$$=5\sec x\tan x.-4\sin x$$

(iv) Let *f* (*x*) = cosec *x*. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x$$

(v) Let $f(x) = 3\cot x + 5\operatorname{cosec} x$.

Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{3\cot(x+h)+5\operatorname{cosec}(x+h)-3\cot x-5\operatorname{cosec} x}{h}$$

$$= 3\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]+5\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec} x\right] \qquad \ldots(1)$$

Now, $\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)\sin x-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]$$

$$= -\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\sin x\cdot\sin(x+h)}\right)$$

$$= -1.\frac{1}{\sin x\cdot\sin(x+0)} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \qquad \ldots(2)$$

$$\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$=\lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$=\left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$=-\operatorname{cosec}x\cot x \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$f'(x)=-3\operatorname{cosec}^2 x-5\operatorname{cosec}x\cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left[5\sin(x+h) - 6\cos(x+h) + 7 - 5\sin x + 6\cos x - 7\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[5\left\{\sin(x+h)-\sin x\right\} - 6\left\{\cos(x+h)-\cos x\right\}\right]$$
$$= 5\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h)-\sin x\right] - 6\lim_{h\to 0}\frac{1}{h}\left[\cos(x+h)-\cos x\right]$$
$$= 5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right] - 6\lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$
$$= 5\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h}{2}\right)\sin\frac{h}{2}\right] - 6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h) - \sin x\sin h}{h}\right]$$
$$= 5\lim_{h\to 0}\left(\cos\left(\frac{2x+h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) - 6\lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h} - \frac{\sin x\sin h}{h}\right]$$
$$= 5\left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\left[\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right] - 6\left[(-\cos x)\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right) - \sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)\right]$$
$$= 5\cos x.1 - 6\left[(-\cos x).(0) - \sin x.1\right]$$
$$= 5\cos x + 6\sin x$$

(vii) Let $f(x) = 2\tan x - 7\sec x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\tan(x+h)-7\sec(x+h)-2\tan x+7\sec x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[2\{\tan(x+h)-\tan x\}-7\{\sec(x+h)-\sec x\}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\tan(x+h)-\tan x\right]-7\lim_{h\to 0}\frac{1}{h}\left[\sec(x+h)-\sec x\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$=2.1.\frac{1}{\cos x\cos x}-7.1\left(\frac{\sin x}{\cos x\cos x}\right)$$

$$=2\sec^2 x-7\sec x\tan x$$

Mathematics

(Chapter – 13) (Limits and Derivatives)
(Class XI)

Miscellaneous Exercise

Question 1:

Find the derivative of the following functions from first principle:

(i) $-x$ (ii) $(-x)^{-1}$

(iii) $\sin(x + 1)$ (iv) $\cos\left(x - \frac{\pi}{8}\right)$

Answer 1:

(i) Let $f(x) = -x$. Accordingly, $f(x+h) = -(x+h)$

By first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$
$$= \lim_{h \to 0} \frac{-x - h + x}{h}$$
$$= \lim_{h \to 0} \frac{-h}{h}$$
$$= \lim_{h \to 0} (-1) = -1$$

(ii) Let $f(x)=(-x)^{-1}=\frac{1}{-x}=\frac{-1}{x}$. Accordingly, $f(x+h)=\frac{-1}{(x+h)}$

By first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}-\left(\frac{-1}{x}\right)\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-1}{x+h}+\frac{1}{x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-x+(x+h)}{x(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{-x+x+h}{x(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{h}{x(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{x(x+h)} \\
&= \frac{1}{x\cdot x}=\frac{1}{x^2}
\end{aligned}
$$

(iii) Let $f(x)$ = sin (x + 1). Accordingly, $f(x+h)=\sin(x+h+1)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+1)-\sin(x+1)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+1+x+1}{2}\right)\sin\left(\frac{x+h+1-x-1}{2}\right)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+h+2}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$
$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+h+2}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$
$$=\lim_{h\to 0}\cos\left(\frac{2x+h+2}{2}\right)\cdot\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$
$$=\cos\left(\frac{2x+0+2}{2}\right)\cdot 1 \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$
$$=\cos(x+1)$$

(iv) Let $f(x)=\cos\left(x-\frac{\pi}{8}\right)$. Accordingly, $f(x+h)=\cos\left(x+h-\frac{\pi}{8}\right)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\cos\left(x+h-\frac{\pi}{8}\right)-\cos\left(x-\frac{\pi}{8}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\frac{\left(x+h-\frac{\pi}{8}+x-\frac{\pi}{8}\right)}{2}\sin\left(\frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2}\right)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[-2\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\sin\frac{h}{2}\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]$$

$$=\lim_{h\to 0}\left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right)\right].\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$

$$=-\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right).1$$

$$=-\sin\left(x-\frac{\pi}{8}\right)$$

Question 2:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (*x* + *a*)

Answer 2:

Let *f*(*x*) = *x* + *a*. Accordingly, $f(x+h)=x+h+a$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{x+h+a-x-a}{h}$$
$$=\lim_{h\to 0}\left(\frac{h}{h}\right)$$
$$=\lim_{h\to 0}(1)$$
$$=1$$

Question 3:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(px+q)\left(\frac{r}{x}+s\right)$

Answer 3: Let $f(x)=(px+q)\left(\frac{r}{x}+s\right)$

$$f'(x)=(px+q)\left(\frac{r}{x}+s\right)'+\left(\frac{r}{x}+s\right)(px+q)'$$
$$=(px+q)\left(rx^{-1}+s\right)'+\left(\frac{r}{x}+s\right)(p)$$
$$=(px+q)\left(-rx^{-2}\right)+\left(\frac{r}{x}+s\right)p$$
$$=(px+q)\left(\frac{-r}{x^2}\right)+\left(\frac{r}{x}+s\right)p$$
$$=\frac{-pr}{x}-\frac{qr}{x^2}+\frac{pr}{x}+ps$$
$$=ps-\frac{qr}{x^2}$$

Question 4:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers)

Answer 4:

Let $f(x)=(ax+b)(cx+d)^2$

By product rule,

$$
\begin{aligned}
f'(x) &= (ax+b)\frac{d}{dx}(cx+d)^2+(cx+d)^2\frac{d}{dx}(ax+b) \\
&= (ax+b)\frac{d}{dx}\left(c^2x^2+2cdx+d^2\right)+(cx+d)^2\frac{d}{dx}(ax+b) \\
&= (ax+b)\left[\frac{d}{dx}\left(c^2x^2\right)+\frac{d}{dx}(2cdx)+\frac{d}{dx}d^2\right]+(cx+d)^2\left[\frac{d}{dx}ax+\frac{d}{dx}b\right] \\
&= (ax+b)\left(2c^2x+2cd\right)+\left(cx+d^2\right)a \\
&= 2c(ax+b)(cx+d)+a(cx+d)^2
\end{aligned}
$$

Question 5:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\dfrac{ax+b}{cx+d}$

Answer 5:

Let $f(x)=\dfrac{ax+b}{cx+d}$

$$
\begin{aligned}
f'(x) &= \frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2} \\
&= \frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2} \\
&= \frac{acx+ad-acx-bc}{(cx+d)^2} \\
&= \frac{ad-bc}{(cx+d)^2}
\end{aligned}
$$

Question 6:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

Answer 6:

$$\text{Let } f(x)=\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{\frac{x+1}{x}}{\frac{x-1}{x}}=\frac{x+1}{x-1}, \text{ where } x\neq 0$$

By quotient rule,

$$f'(x)=\frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{x-1-x-1}{(x-1)^2},\ x\neq 0,\ 1$$

$$=\frac{-2}{(x-1)^2},\ x\neq 0,\ 1$$

Question 7:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{1}{ax^2+bx+c}$

Answer 7:

Let $f(x)=\frac{1}{ax^2+bx+c}$

By quotient rule,

$$f'(x)=\frac{\left(ax^2+bx+c\right)\frac{d}{dx}(1)-\frac{d}{dx}\left(ax^2+bx+c\right)}{\left(ax^2+bx+c\right)^2}$$

$$=\frac{\left(ax^2+bx+c\right)(0)-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

$$=\frac{-(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

Question 8:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{ax+b}{px^2+qx+r}$

Answer 8:

Let $f(x)=\frac{ax+b}{px^2+qx+r}$

By quotient rule,

$$f'(x)=\frac{\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{\left(px^2+qx+r\right)(a)-(ax+b)(2px+q)}{\left(px^2+qx+r\right)^2}$$

$$=\frac{apx^2+aqx+ar-2apx^2-aqx-2bpx-bq}{\left(px^2+qx+r\right)^2}$$

$$=\frac{-apx^2-2bpx+ar-bq}{\left(px^2+qx+r\right)^2}$$

Question 9:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{px^2+qx+r}{ax+b}$

Answer 9:

Let $f(x)=\frac{px^2+qx+r}{ax+b}$

By quotient rule,

$$f'(x)=\frac{(ax+b)\frac{d}{dx}\left(px^2+qx+r\right)-\left(px^2+qx+r\right)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$=\frac{(ax+b)(2px+q)-\left(px^2+qx+r\right)(a)}{(ax+b)^2}$$

$$=\frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$=\frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

Question 10:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{a}{x^4}-\frac{b}{x^2}+\cos x$

Answer 10:

$$\text{Let } f(x)=\frac{a}{x^4}-\frac{b}{x^2}+\cos x$$

$$f'(x)=\frac{d}{dx}\left(\frac{a}{x^4}\right)-\frac{d}{dx}\left(\frac{b}{x^2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\frac{d}{dx}\left(x^{-4}\right)-b\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}(\cos x)$$

$$=a\left(-4x^{-5}\right)-b\left(-2x^{-3}\right)+(-\sin x) \qquad \left[\frac{d}{dx}\left(x^n\right)=nx^{n-1}\text{and}\frac{d}{dx}(\cos x)=-\sin x\right]$$

$$=\frac{-4a}{x^5}+\frac{2b}{x^3}-\sin x$$

Question 11:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $4\sqrt{x}-2$

Answer 11:

$$\text{Let } f(x)=4\sqrt{x}-2$$

$$f'(x)=\frac{d}{dx}\left(4\sqrt{x}-2\right)=\frac{d}{dx}\left(4\sqrt{x}\right)-\frac{d}{dx}(2)$$

$$=4\frac{d}{dx}\left(x^{\frac{1}{2}}\right)-0=4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$=\left(2x^{-\frac{1}{2}}\right)=\frac{2}{\sqrt{x}}$$

Question 12:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax + b)^n$

Answer 12:

Let $f(x) = (ax+b)^n$. Accordingly, $f(x+h) = \{a(x+h)+b\}^n = (ax+ah+b)^n$

By first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(ax+ah+b)^n-(ax+b)^n}{h}$$

$$= \lim_{h\to 0}\frac{(ax+b)^n\left(1+\frac{ah}{ax+b}\right)^n-(ax+b)^n}{h}$$

$$= (ax+b)^n \lim_{h\to 0}\frac{\left(1+\frac{ah}{ax+b}\right)^n-1}{h}$$

$$= (ax+b)^n \lim_{h\to 0}\frac{1}{n}\left[\left\{1+n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)}{\underline{|2}}\left(\frac{ah}{ax+b}\right)^2+\ldots\right\}-1\right]$$

(Using binomial theorem)

$$= (ax+b)^n \lim_{h\to 0}\frac{1}{h}\left[n\left(\frac{ah}{ax+b}\right)+\frac{n(n-1)a^2h^2}{\underline{|2}(ax+b)^2}+\ldots\text{(Terms containing higher degrees of } h)\right]$$

$$= (ax+b)^n \lim_{h\to 0}\left[\frac{na}{(ax+b)}+\frac{n(n-1)a^2h}{\underline{|2}(ax+b)^2}+\ldots\right]$$

$$= (ax+b)^n\left[\frac{na}{(ax+b)}+0\right]$$

$$= na\frac{(ax+b)^n}{(ax+b)}$$

$$= na(ax+b)^{n-1}$$

Question 13:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax + b)^n (cx + d)^m$

Answer 13: Let $f(x)=(ax+b)^n(cx+d)^m$

$$f'(x)=(ax+b)^n\frac{d}{dx}(cx+d)^m+(cx+d)^m\frac{d}{dx}(ax+b)^n \qquad ...(1)$$

Now, let $f_1(x)=(cx+d)^m$

$$f_1(x+h)=(cx+ch+d)^m$$

$$f_1'(x)=\lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h}$$

$$=\lim_{h\to 0}\frac{(cx+ch+d)^m-(cx+d)^m}{h}$$

$$=(cx+d)^m\lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{ch}{cx+d}\right)^m-1\right]$$

$$=(cx+d)^m\lim_{h\to 0}\frac{1}{h}\left[\left(1+\frac{mch}{(cx+d)}+\frac{m(m-1)}{2}\frac{(c^2h^2)}{(cx+d)^2}+...\right)-1\right]$$

$$=(cx+d)^m\lim_{h\to 0}\frac{1}{h}\left[\frac{mch}{(cx+d)}+\frac{m(m-1)c^2h^2}{2(cx+d)^2}+...(\text{Terms containing higher degrees of } h)\right]$$

$$=(cx+d)^m\lim_{h\to 0}\left[\frac{mc}{(cx+d)}+\frac{m(m-1)c^2h}{2(cx+d)^2}+...\right]$$

$$=(cx+d)^m\left[\frac{mc}{cx+d}+0\right]$$

$$=\frac{mc(cx+d)^m}{(cx+d)}$$

$$=mc(cx+d)^{m-1}$$

$$\frac{d}{dx}(cx+d)^m=mc(cx+d)^{m-1} \qquad ...(2)$$

Similarly, $\frac{d}{dx}(ax+b)^n=na(ax+b)^{n-1}$ ...(3)

Therefore, from (1), (2), and (3), we obtain

$$f'(x)=(ax+b)^{n}\left\{mc(cx+d)^{m-1}\right\}+(cx+d)^{m}\left\{na(ax+b)^{n-1}\right\}$$
$$=(ax+b)^{n-1}(cx+d)^{m-1}\left[mc(ax+b)+na(cx+d)\right]$$

Question 14:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): sin (*x* + *a*)

Answer 14:

Let $f(x)=\sin(x+a)$, therefore $f(x+h)=\sin(x+h+a)$

By first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$=\lim_{h\to 0}\frac{\sin(x+h+a)-\sin(x+a)}{h}$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$
$$=\lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$
$$=\lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right)\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \qquad \left[\text{As } h\to 0\Rightarrow\frac{h}{2}\to 0\right]$$
$$=\cos\left(\frac{2x+2a}{2}\right)\times 1 \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$
$$=\cos(x+a)$$

Question 15:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): cosec *x* cot *x*

Answer 15:

Let $f(x) = \operatorname{cosec} x \cot x$

By product rule,

$$f'(x) = \operatorname{cosec} x(\cot x)' + \cot x(\operatorname{cosec} x)' \qquad ...(1)$$

Let $f_1(x) = \cot x$. Accordingly, $f_1(x+h) = \cot(x+h)$

By first principle,

$$\begin{aligned}
f_1'(x) &= \lim_{h\to 0}\frac{f_1(x+h)-f_1(x)}{h} \\
&= \lim_{h\to 0}\frac{\cot(x+h)-\cot x}{h} \\
&= \lim_{h\to 0}\frac{1}{h}\left(\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x\cos(x+h)-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right] \\
&= \frac{1}{\sin x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin(x+h)}\right] \\
&= \frac{-1}{\sin x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\sin(x+h)}\right) \\
&= \frac{-1}{\sin x}.1.\left(\frac{1}{\sin(x+0)}\right) \\
&= \frac{-1}{\sin^2 x} \\
&= -\operatorname{cosec}^2 x
\end{aligned}$$

$$\therefore (\cot x)' = -\operatorname{cosec}^2 x \qquad ...(2)$$

Now, let $f_2(x)$ = cosec x. Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By first principle,

$$f_2'(x) = \lim_{h \to 0} \frac{f_2(x+h) - f_2(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\operatorname{cosec}(x+h) - \operatorname{cosec} x\right]$$

From (1), (2), and (3), we obtain

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0} \frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{1}{\sin x} \cdot \lim_{h \to 0}\left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}\right]$$

$$= \frac{-1}{\sin x} \cdot \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \to 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)}$$

$$= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)}$$

$$= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= -\operatorname{cosec} x \cdot \cot x$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x \qquad \ldots(3)$$

$$f'(x) = \operatorname{cosec} x\left(-\operatorname{cosec}^2 x\right) + \cot x\left(-\operatorname{cosec} x \cot x\right)$$
$$= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

Question 16:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\cos x}{1+\sin x}$

Answer 16:

Let $f(x) = \frac{\cos x}{1+\sin x}$

By quotient rule,

$$f'(x) = \frac{(1+\sin x)\frac{d}{dx}(\cos x) - (\cos x)\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$
$$= \frac{(1+\sin x)(-\sin x) - (\cos x)(\cos x)}{(1+\sin x)^2}$$
$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$
$$= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$
$$= \frac{-\sin x - 1}{(1+\sin x)^2}$$
$$= \frac{-(1+\sin x)}{(1+\sin x)^2}$$
$$= \frac{-1}{(1+\sin x)}$$

Question 17:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sin x+\cos x}{\sin x-\cos x}$

Answer 17:

Let $f(x)=\frac{\sin x+\cos x}{\sin x-\cos x}$

By quotient rule,

$$f'(x)=\frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x)-(\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$=\frac{(\sin x-\cos x)(\cos x-\sin x)-(\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$=\frac{-(\sin x-\cos x)^2-(\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$=\frac{-\left[\sin^2 x+\cos^2 x-2\sin x\cos x+\sin^2 x+\cos^2 x+2\sin x\cos x\right]}{(\sin x-\cos x)^2}$$

$$=\frac{-[1+1]}{(\sin x-\cos x)^2}$$

$$=\frac{-2}{(\sin x-\cos x)^2}$$

Question 18:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sec x-1}{\sec x+1}$

Answer 18:

Let $f(x)=\frac{\sec x-1}{\sec x+1}$ $\quad f(x)=\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}=\frac{1-\cos x}{1+\cos x}$

By quotient rule,

$$f'(x)=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)(\sin x)-(1-\cos x)(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\sin x+\cos x\sin x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{\left(1+\frac{1}{\sec x}\right)^2}=\frac{2\sin x}{\frac{(\sec x+1)^2}{\sec^2 x}}$$

$$=\frac{2\sin x\sec^2 x}{(\sec x+1)^2}$$

$$=\frac{\frac{2\sin x}{\cos x}\sec x}{(\sec x+1)^2}$$

$$=\frac{2\sec x\tan x}{(\sec x+1)^2}$$

Question 19:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\sin^n x$

Answer 19:

Let $y = \sin^n x$.

Accordingly, for $n = 1$, $y = \sin x$.

$$\therefore \frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx}\sin x = \cos x$$

For $n = 2$, $y = \sin^2 x$.

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d}{dx}(\sin x \sin x) \\ &= (\sin x)' \sin x + \sin x (\sin x)' && \text{[By Leibnitz product rule]} \\ &= \cos x \sin x + \sin x \cos x \\ &= 2\sin x \cos x && \text{...(1)}\end{aligned}$$

For $n = 3$, $y = \sin^3 x$.

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d}{dx}(\sin x \sin^2 x) \\ &= (\sin x)' \sin^2 x + \sin x (\sin^2 x)' && \text{[By Leibnitz product rule]} \\ &= \cos x \sin^2 x + \sin x (2\sin x \cos x) && \text{[Using (1)]} \\ &= \cos x \sin^2 x + 2\sin^2 x \cos x \\ &= 3\sin^2 x \cos x\end{aligned}$$

We assert that $\frac{d}{dx}(\sin^n x) = n\sin^{(n-1)} x \cos x$

Let our assertion be true for $n = k$.

i.e., $\frac{d}{dx}\left(\sin^{k} x\right)=k \sin^{(k-1)} x \cos x$ ...(2)

Consider

$$\frac{d}{dx}\left(\sin^{k+1} x\right)=\frac{d}{dx}\left(\sin x \sin^{k} x\right)$$

$$=(\sin x)' \sin^{k} x+\sin x\left(\sin^{k} x\right)' \quad \text{[By Leibnitz product rule]}$$

$$=\cos x \sin^{k} x+\sin x\left(k \sin^{(k-1)} x \cos x\right) \quad \text{[Using (2)]}$$

$$=\cos x \sin^{k} x+k \sin^{k} x \cos x$$

$$=(k+1) \sin^{k} x \cos x$$

Thus, our assertion is true for $n = k + 1$.

Hence, by mathematical induction, $\frac{d}{dx}\left(\sin^{n} x\right)=n \sin^{(n-1)} x \cos x$

Question 20:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{a+b \sin x}{c+d \cos x}$

Answer 20:

Let $f(x)=\frac{a+b \sin x}{c+d \cos x}$

By quotient rule,

$$f'(x)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

$$=\frac{cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd\left(\cos^2 x+\sin^2 x\right)}{(c+d\cos x)^2}$$

$$=\frac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

Question 21:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{\sin(x+a)}{\cos x}$

Answer 21:

Let $f(x)=\frac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x)=\frac{\cos x\frac{d}{dx}[\sin(x+a)]-\sin(x+a)\frac{d}{dx}\cos x}{\cos^2 x}$$

$$f'(x)=\frac{\cos x\frac{d}{dx}[\sin(x+a)]-\sin(x+a)(-\sin x)}{\cos^2 x} \quad \ldots \text{(i)}$$

Let $g(x)=\sin(x+a)$. Accordingly, $g(x+h)=\sin(x+h+a)$

By first principle,

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\sin(x+h+a)-\sin(x+a)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{x+h+a+x+a}{2}\right)\sin\left(\frac{x+h+a-x-a}{2}\right)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[2\cos\left(\frac{2x+2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+2a+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\}\right]$$

$$= \lim_{h\to 0}\cos\left(\frac{2x+2a+h}{2}\right).\lim_{\frac{h}{2}\to 0}\left\{\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right\} \qquad \left[\text{As } h\to 0 \Rightarrow \frac{h}{2}\to 0\right]$$

$$= \left(\cos\frac{2x+2a}{2}\right)\times 1 \qquad \left[\lim_{h\to 0}\frac{\sin h}{h}=1\right]$$

$$= \cos(x+a) \qquad \ldots \text{(ii)}$$

From (i) and (ii), we obtain

$$f'(x) = \frac{\cos x\cdot\cos(x+a)+\sin x\sin(x+a)}{\cos^2 x}$$

$$= \frac{\cos(x+a-x)}{\cos^2 x}$$

$$= \frac{\cos a}{\cos^2 x}$$

Question 22:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $x^4 (5 \sin x - 3 \cos x)$

Answer 22:

Let

$f(x) = x^4 (5\sin x - 3\cos x)$ By

product rule,

$$f'(x) = x^4 \frac{d}{dx}(5\sin x - 3\cos x) + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$

$$= x^4 \left[5\frac{d}{dx}(\sin x) - 3\frac{d}{dx}(\cos x)\right] + (5\sin x - 3\cos x)\frac{d}{dx}(x^4)$$

$$= x^4 [5\cos x - 3(-\sin x)] + (5\sin x - 3\cos x)(4x^3)$$

$$= x^3 [5x\cos x + 3x\sin x + 20\sin x - 12\cos x]$$

Question 23:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(x^2 + 1) \cos x$

Answer 23:

Let $f(x) = (x^2 + 1)\cos x$

By product rule,

$$f'(x) = (x^2 + 1)\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2 + 1)$$

$$= (x^2 + 1)(-\sin x) + \cos x(2x)$$

$$= -x^2 \sin x - \sin x + 2x\cos x$$

Question 24:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(ax^2 + \sin x)(p + q\cos x)$

Answer 24:

Let
$f(x) = (ax^2 + \sin x)(p + q\cos x)$ By
product rule,

$$f'(x) = (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x) + (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x)$$
$$= (ax^2 + \sin x)(-q\sin x) + (p + q\cos x)(2ax + \cos x)$$
$$= -q\sin x(ax^2 + \sin x) + (p + q\cos x)(2ax + \cos x)$$

Question 25:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $(x + \cos x)(x - \tan x)$

Answer 25:

Let $f(x) = (x + \cos x)(x - \tan x)$

By product rule,

$$f'(x) = (x + \cos x)\frac{d}{dx}(x - \tan x) + (x - \tan x)\frac{d}{dx}(x + \cos x)$$
$$= (x + \cos x)\left[\frac{d}{dx}(x) - \frac{d}{dx}(\tan x)\right] + (x - \tan x)(1 - \sin x)$$
$$= (x + \cos x)\left[1 - \frac{d}{dx}\tan x\right] + (x - \tan x)(1 - \sin x) \qquad \text{... (i)}$$

Let $g(x) = \tan x$. Accordingly, $g(x + h) = \tan(x + h)$

By first principle,

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right) \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right] \\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)}\right] \\
&= \frac{1}{\cos x}.\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)}\right) \\
&= \frac{1}{\cos x}.1.\frac{1}{\cos(x+0)} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x \qquad \ldots \text{(ii)}
\end{aligned}
$$

Therefore, from (i) and (ii), we obtain

$$
\begin{aligned}
f'(x) &= (x+\cos x)(1-\sec^2 x)+(x-\tan x)(1-\sin x) \\
&= (x+\cos x)(-\tan^2 x)+(x-\tan x)(1-\sin x) \\
&= -\tan^2 x(x+\cos x)+(x-\tan x)(1-\sin x)
\end{aligned}
$$

Question 26:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{4x+5\sin x}{3x+7\cos x}$

Answer 26:

Let $f(x)=\frac{4x+5\sin x}{3x+7\cos x}$

By quotient rule,

$$f'(x)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right]-(4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$=\frac{12x+15x\cos x+28\cos x+35\cos^2 x-12x+28x\sin x-15\sin x+35\sin^2 x}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28\cos x+28x\sin x-15\sin x+35\left(\cos^2 x+\sin^2 x\right)}{(3x+7\cos x)^2}$$

$$=\frac{35+15x\cos x+28\cos x+28x\sin x-15\sin x}{(3x+7\cos x)^2}$$

Question 27:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers):

$$\frac{x^2\cos\left(\frac{\pi}{4}\right)}{\sin x}$$

Answer 27:

Let $f(x)=\dfrac{x^2\cos\left(\dfrac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x)=\cos\frac{\pi}{4}.\left[\frac{\sin x\dfrac{d}{dx}\left(x^2\right)-x^2\dfrac{d}{dx}(\sin x)}{\sin^2 x}\right]$$

$$=\cos\frac{\pi}{4}.\left[\frac{\sin x\cdot 2x-x^2\cos x}{\sin^2 x}\right]$$

$$=\frac{x\cos\dfrac{\pi}{4}\left[2\sin x-x\cos x\right]}{\sin^2 x}$$

Question 28:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\dfrac{x}{1+\tan x}$

Answer 28:

Let $f(x)=\dfrac{x}{1+\tan x}$

$$f'(x)=\frac{(1+\tan x)\dfrac{d}{dx}(x)-x\dfrac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$f'(x)=\frac{(1+\tan x)-x\cdot\dfrac{d}{dx}(1+\tan x)}{(1+\tan x)^2} \qquad \ldots\text{(i)}$$

Let $g(x)=1+\tan x$. Accordingly, $g(x+h)=1+\tan(x+h)$.

By first principle,

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$
$$=\lim_{h\to 0}\left[\frac{1+\tan(x+h)-1-\tan x}{h}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$
$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$
$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$
$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$
$$\Rightarrow\frac{d}{dx}(1+\tan x)=\sec^2 x \qquad \text{... (ii)}$$

From (i) and (ii), we obtain

$$f'(x)=\frac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$$

Question 29:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): (x + sec x) (x – tan x)

Answer 29:

Let $f(x)=(x+\sec x)(x-\tan x)$

By product rule,

$$f'(x)=(x+\sec x)\frac{d}{dx}(x-\tan x)+(x-\tan x)\frac{d}{dx}(x+\sec x)$$
$$=(x+\sec x)\left[\frac{d}{dx}(x)-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[\frac{d}{dx}(x)+\frac{d}{dx}\sec x\right]$$
$$=(x+\sec x)\left[1-\frac{d}{dx}\tan x\right]+(x-\tan x)\left[1+\frac{d}{dx}\sec x\right] \quad \text{... (i)}$$

Let $f_1(x)=\tan x$, $f_2(x)=\sec x$

Accordingly, $f_1(x+h)=\tan(x+h)$ and $f_2(x+h)=\sec(x+h)$

$$f_1'(x)=\lim_{h\to 0}\left(\frac{f_1(x+h)-f_1(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

$$=\lim_{h\to 0}\left[\frac{\tan(x+h)-\tan x}{h}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h)\cos x-\sin x\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin h}{\cos(x+h)\cos x}\right]$$

$$=\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\cos(x+h)\cos x}\right)$$

$$=1\times\frac{1}{\cos^2 x}=\sec^2 x$$

$$\Rightarrow \frac{d}{dx}\tan x=\sec^2 x \qquad \text{... (ii)}$$

$$f_2'(x)=\lim_{h\to 0}\left(\frac{f_2(x+h)-f_2(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\sec(x+h)-\sec x}{h}\right)$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos(x+h)\cos x}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\cdot\sin\left(\frac{-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right)\left\{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\cos(x+h)}\right]$$

$$=\sec x.\frac{\left\{\lim_{h\to 0}\sin\left(\frac{2x+h}{2}\right)\right\}\left\{\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right\}}{\lim_{h\to 0}\cos(x+h)}$$

$$=\sec x.\frac{\sin x.1}{\cos x}$$

$$\Rightarrow\frac{d}{dx}\sec x=\sec x\tan x \qquad \text{.. (iii)}$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x+\sec x)(1-\sec^2 x)+(x-\tan x)(1+\sec x\tan x)$$

Question 30:

Find the derivative of the following functions (it is to be understood that *a*, *b*, *c*, *d*, *p*, q, *r* and *s* are fixed non-zero constants and *m* and *n* are integers): $\frac{x}{\sin^n x}$

Answer 30:

Let $f(x) = \frac{x}{\sin^n x}$

By quotient rule,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

It can be easily shown that $\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$

Therefore,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

$$= \frac{\sin^n x.1 - x\left(n \sin^{n-1} x \cos x\right)}{\sin^{2n} x}$$

$$= \frac{\sin^{n-1} x(\sin x - nx\cos x)}{\sin^{2n} x}$$

$$= \frac{\sin x - nx\cos x}{\sin^{n+1} x}$$

Mathematics

(Chapter – 13) (Limits and Derivatives)

(Class – XI)

Exercise 13.2 (Supplementary)

$Evaluate\ the\ following\ limits, if\ exist.$

Question 1: $\lim\limits_{x\to 0} \frac{e^{4x}-1}{x}$

Answer 1: $\lim\limits_{x\to 0} \frac{e^{4x}-1}{x}$

$= \lim\limits_{x\to 0} \frac{e^{4x}-1}{4x} \times 4$

$= \lim\limits_{y\to 0} \frac{e^{y}-1}{y} \times 4$ [$Where\ y = 4x$]

= 1×4 [Using $\lim\limits_{y\to 0} \frac{e^{y}-1}{y} = 1$]

= 4

Question 2: $\lim\limits_{x\to 0} \frac{e^{2+x}-e^2}{x}$

Answer 2: $\lim\limits_{x\to 0} \frac{e^{2+x}-e^2}{x}$

$= \lim\limits_{x\to 0} \frac{e^2(e^x-1)}{x}$

$= e^2 \times 1$ [Using $\lim\limits_{x\to 0} \frac{e^{x}-1}{x} = 1$]

$= e^2$

Question 3: $\lim\limits_{x\to 5} \frac{e^x - e^5}{x - 5}$

Answer 3: $\lim\limits_{x\to 5} \frac{e^x - e^5}{x-5}$

$Put\ x = 5 + h, then\ as\ x \to 5 \Longrightarrow h \longrightarrow 0. Therefore$

$$\lim_{x\to 5}\frac{e^x-e^5}{x-5} = \lim_{h\to 0}\frac{e^{5+h}-e^5}{h}$$

$$= \lim_{h\to 0}\frac{e^5(e^h-1)}{h}$$

$$= e^5 \times 1 \qquad \left[\text{Using } \lim_{h\to 0}\frac{e^h-1}{h} = 1\right]$$

$$= e^5$$

Question 4: $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$

Answer 4: $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$

$$= \lim_{x\to 0}\frac{e^{\sin x}-1}{x} \times \frac{\sin x}{\sin x}$$

$$= \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x} \times \frac{\sin x}{x}$$

$$= \lim_{y\to 0}\frac{e^y-1}{y} \times \lim_{x\to 0}\frac{\sin x}{x} \qquad [Where\ y = \sin x]$$

$$= 1 \times 1 \qquad \left[\text{Using } \lim_{y\to 0}\frac{e^y-1}{y} = 1\ and\ \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$$= 1$$

Question 5: $\lim_{x\to 3}\frac{e^x-e^3}{x-3}$

Answer 5: $\lim_{x\to 3}\frac{e^x-e^3}{x-3}$

$Put\ x = 3+h, then\ as\ x \to 3 \Longrightarrow h \longrightarrow 0. Therefore$

$$\lim_{x \to 3} \frac{e^x - 3}{x-3} = \lim_{h \to 0} \frac{e^{3+h} - e^3}{h}$$

$$= \lim_{h \to 0} \frac{e^3(e^h - 1)}{h}$$

$$= e^3 \times 1 \qquad \left[\text{Using } \lim_{h \to 0} \frac{e^h - 1}{h} = 1\right]$$

$$= e^3$$

Question 6: $\lim_{x \to 0} \frac{x\,(e^{\,x} - 1)}{1 - \cos x}$

Answer 6: $\lim_{x \to 0} \frac{x\,(e^{\,x} - 1)}{1 - \cos x}$

$$= \lim_{x \to 0} \frac{x\,(e^{\,x} - 1)}{1 - \cos x} \times \frac{1 + \cos x}{1 + \cos x} \times \frac{x}{x}$$

$$= \lim_{x \to 0} \frac{(e^{\,x} - 1)}{x} \times \frac{1 + \cos x}{1} \times \frac{x^2}{1 - cos^2 x}$$

$$= \lim_{x \to 0} \frac{(e^{\,x} - 1)}{x} \times \lim_{x \to 0} \frac{1 + \cos x}{1} \times \lim_{x \to 0} \frac{x^2}{sin^2 x}$$

$$= \lim_{x \to 0} \frac{(e^{\,x} - 1)}{x} \times \lim_{x \to 0} \frac{1 + \cos x}{1} \times \lim_{x \to 0} \frac{1}{\left(\frac{\sin x}{x}\right)^2}$$

$$= 1 \times (1 + 1) \times \frac{1}{1^1} \qquad \left[\text{Using } \lim_{x \to 0} \frac{e^x - 1}{x} = 1 \text{ and } \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= 2$$

Question 7: $\lim_{x \to 0} \frac{log_e\,(1 + 2x)}{x}$

Answer 7: $\lim_{x \to 0} \frac{log_e\,(1 + 2x)}{x}$

$$= \lim_{x \to 0} \frac{log_e\,(1 + 2x)}{2x} \times 2$$

$= \lim\limits_{y \to 0} \frac{\log_e(1+y)}{y} \times 2$ $\quad$ $[Where\ y = 2x]$

$= 1 \times 2$ $\quad$ $[Using\ \lim\limits_{y \to 0} \frac{\log_e(1+y)}{y} = 1]$

$= 2$

Question 8: $\lim\limits_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x}$

Answer 8: $\lim\limits_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x}$

$= \lim\limits_{x \to 0} \frac{\log(1+x^3)}{\sin^3 x} \times \frac{x^3}{x^3}$

$= \lim\limits_{x \to 0} \frac{\log(1+x^3)}{x^3} \times \frac{x^3}{\sin^3 x}$

$= \lim\limits_{y \to 0} \frac{\log(1+y)}{y} \times \lim\limits_{x \to 0} \frac{1}{\left(\frac{\sin x}{x}\right)^3}$ $\quad$ $[Where\ y = x^3]$

$= 1 \times \frac{1}{1^3}$ $\quad$ $[Using\ \lim\limits_{y \to 0} \frac{\log(1+y)}{y} = 1\ and\ \lim\limits_{x \to 0} \frac{\sin x}{x} = 1]$

$= 1$