

CBSE Test Paper 01
Chapter 8 Application of Integrals

1. The area bounded by the curves $y^2 = 20x$ and $x^2 = 16y$ is equal to
 - a. $\frac{320}{3}$ sq. units
 - b. 80π sq. units
 - c. none of these
 - d. 100π sq. units
2. The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to the parabola at the point $(2, 3)$ and the x – axis is equal to
 - a. none of these
 - b. 6 sq. units
 - c. 9 sq. units
 - d. 12 sq. units
3. The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and the x – axis in the first quadrant is
 - a. 36
 - b. 18
 - c. 9
 - d. none of these
4. If the area cut off from a parabola by any double ordinate is k times the corresponding rectangle contained by that double ordinate and its distance from the vertex, then k is equal to
 - a. $\frac{2}{3}$
 - b. 3
 - c. $\frac{1}{3}$
 - d. $\frac{3}{2}$
5. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$

and $x = \frac{\pi}{2}$ is equal to

- a. $2(\sqrt{2} + 1)$ sq. units
- b. $2(\sqrt{2} - 1)$ sq. units
- c. $(4\sqrt{2} - 1)$ sq. units
- d. $(4\sqrt{2} + 1)$ sq. units

6. The area of the bounded by the lines $y = 2$, $x = 1$, $x = a$ and the curve $y = f(x)$, which cuts the last two lines above the first line for all $a \geq 1$, is equal to $\frac{2}{3} [(2a)^{3/2} - 3a + 3 - 2\sqrt{2}]$. Find $f(x)$
7. Let $f(x)$ be a continuous function such that the area bounded by the curve $y=f(x)$, x-axis and the lines $x=0$ and $x=a$ is $\frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a$, then find $f(\frac{\pi}{2})$.
8. Find the area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive x-axis.
9. Calculate the area of the region enclosed between the circles: $x^2 + y^2 = 16$ and $(x + 4)^2 + y^2 = 16$.
10. Using integration, find the area of region bounded by the triangle whose vertices are $(-1, 0)$, $(1, 3)$ and $(3, 2)$.
11. Find the area of the region $\{(x, y) ; x^2 \leq y \leq x\}$.
12. Evaluate $\lim_{x \rightarrow \infty} \left(\frac{x^x}{x!} \right)^{1/x}$.
13. Evaluate $\lim_{x \rightarrow \infty} \left[\frac{1}{x} + \frac{x^2}{(x+1)^3} + \frac{x^2}{(x+2)^3} + \dots + \frac{1}{8x} \right]$.
14. Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y=x + 2$.
15. Using integration, find the area of the region enclosed between the two circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

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Solution

1. (a) $\frac{320}{3}$ sq. units

Explanation: Eliminating y, we get: $x^4 = 256 \times 20x$

$$\Rightarrow x = 0, x = 8(10)^{\frac{1}{3}}$$

Required area:

$$\begin{aligned} &= \int_0^{8(10)^{\frac{1}{3}}} \left(\sqrt{20x} - \frac{x^2}{16} \right) dx \\ &= \frac{640}{3} - \frac{320}{3} = \frac{320}{3} \text{ sq units} \end{aligned}$$

2. (c) 9 sq. units

Explanation: Given parabola is: $(y - 2)^2 = x - 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2(y-2)}$

When y= 3, x= 2

$$\therefore \frac{dy}{dx} = \frac{1}{2}$$

Therefore, tangent at (2, 3) is $y - 3 = \frac{1}{2} (x - 2)$. i.e. $x - 2y + 4 = 0$. therefore required area

$$\text{is: } \int_0^3 (y - 2)^2 + 1 dy - \int_0^3 (2y - 4) dy = \left[\frac{(y-2)^3}{3} + y \right]_0^3 - [y^2 - 4y]_0^3 = 9$$

3. (c) 9

Explanation: Required area: $\int_0^9 \sqrt{x} dx - \int_3^9 \left(\frac{x-3}{2} \right) dx$

$$= \left[\frac{x^{\frac{3}{2}}}{3/2} \right]_0^9 - \frac{1}{2} \left[\frac{x^2}{2} - 3x \right]_3^9 = 9 \text{ sq. units}$$

4. (a) $\frac{2}{3}$

Explanation: Required area: $2 \int_0^a \sqrt{4ax} dx$

$$= k\alpha(2\sqrt{4a\alpha})$$

$$= \frac{8\sqrt{a}}{3}\alpha^{\frac{3}{2}}$$

$$= 4\sqrt{a}k\alpha^{\frac{3}{2}} \Rightarrow k = \frac{2}{3}$$

5. (b) $2(\sqrt{2} - 1)$ sq. units

Explanation: Required area = $\int_0^{\frac{\pi}{2}} |\sin x - \cos x| dx$

$$\begin{aligned}
 &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x) dx \\
 &= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
 &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0 + 1) - \left\{ 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \right\} \\
 &= \frac{4}{\sqrt{2}} - 2 = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)
 \end{aligned}$$

6. we are given,

$$\int_a^1 [f(x) - 2] dx = \frac{2}{3} [(2a)^{3/2} - 3a + 3 - 2\sqrt{2}]$$

Differentiating w.r.t a, we get

$$f(a) - 2 = \frac{2}{3} \left[\frac{3}{2} \sqrt{2a} \cdot 2 - 3 \right]$$

$$f(a) = 2\sqrt{2a}, a \geq 1$$

$$\therefore f(x) = 2\sqrt{2x}, x \geq 1$$

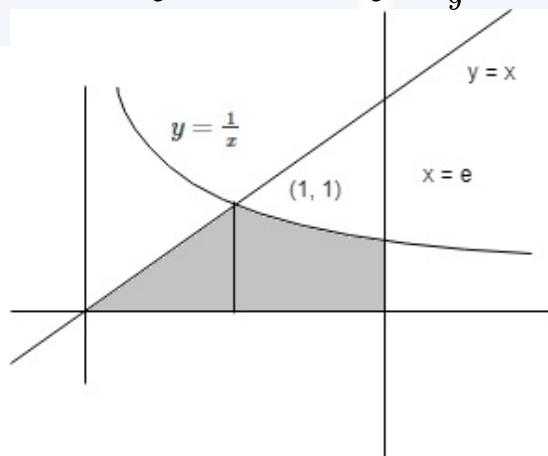
7. we have, $\int_0^a f(x) dx = \frac{a^2}{2} + \frac{a}{2} \sin a + \frac{\pi}{2} \cos a$

Differentiating w.r.t a, we get,

$$f(a) = a + \frac{1}{2} (\sin a + a \cos a) - \frac{\pi}{2} \sin a$$

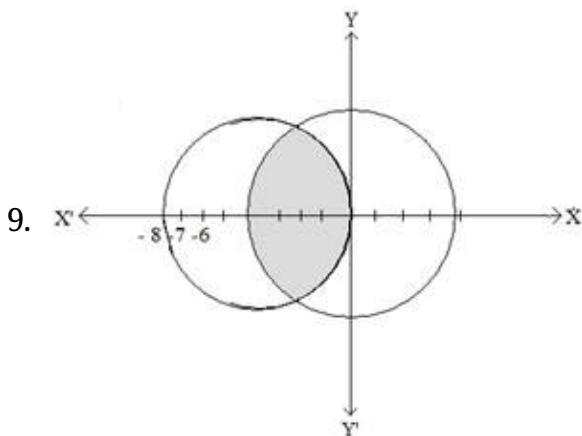
$$\text{put } a = \frac{\pi}{2}, f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \frac{1}{2} - \frac{\pi}{2} = \frac{1}{2}$$

8. We have $y = 4x^2$ and $y = \frac{1}{9}x^2$



$$\text{Required area} = 2 \int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2} \right) dy$$

$$\begin{aligned}
 &= 2 \left(\frac{5y}{2} \frac{\sqrt{y}}{3/2} \right)_0^2 \\
 &= 2 \cdot \frac{5}{3} 2\sqrt{2} = \frac{20\sqrt{2}}{3}
 \end{aligned}$$

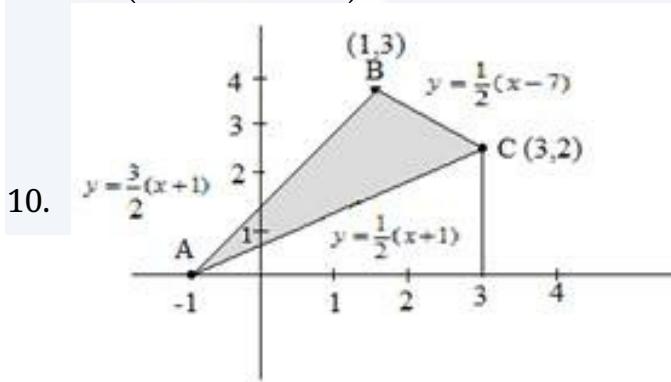


$$x^2 + y^2 = 16$$

$$(x + 4)^2 + y^2 = 16$$

Intersecting at $x = -2$

$$\begin{aligned} \text{Area} &= 4 \int_{-4}^{-2} \sqrt{16 - x^2} dx \\ &= 4 \left[\int_{-4}^{-2} \sqrt{4^2 - x^2} dx \right] = 4 \left[\frac{x}{2} \sqrt{1 - x^2} + \frac{4^2}{2} \sin^{-1} \frac{x}{4} \right]_{-4}^{-2} \\ &= 4 \left[\left(-2\sqrt{3} - \frac{4\pi}{3} \right) - \left(-4\pi \right) \right] \\ &= \left(-8\sqrt{3} + \frac{32\pi}{3} \right) \end{aligned}$$



A (-1, 0) B (1, 3) C (3, 2)

Equation of AB

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$y - 0 = \frac{3-0}{1+1}(x+1)$$

$$y = \frac{3}{2}(x+1)$$

Similarly,

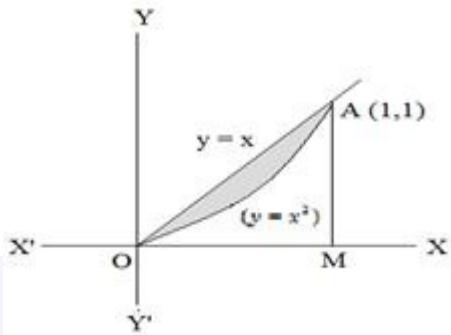
$$\text{Equation of BC } y = \frac{-1}{2}(x-7)$$

$$\text{Equation of AC } y = \frac{1}{2}(x+1)$$

$$\text{Area } \Delta ABC = \int_{-1}^1 \frac{3}{2}(x+1) dx + \int_1^3 \frac{1}{2}(x-7) dx - \int_{-1}^3 \frac{1}{2}(x+1) dx$$

$$\begin{aligned}
&= \frac{3}{2} \left[\frac{x^2}{2} + x \right]_{-1}^1 + \frac{1}{2} \left[7x - \frac{x^2}{2} \right]_1^3 - \left[\frac{x^2}{2} + x \right]_{-1}^3 \\
&= \frac{3}{2} \left[\left(\frac{1}{2} + 1 \right) - \left(\frac{1}{2} - 1 \right) \right] + \frac{1}{2} \left[\left(21 - \frac{9}{2} \right) - \left(7 - \frac{1}{2} \right) \right] \\
&\quad - \frac{1}{2} \left[\left(\frac{9}{2} + 3 \right) - \left(\frac{1}{2} - 1 \right) \right] \\
&= \frac{3}{2}(2) + \frac{1}{2}(10) - \frac{1}{2}(8) = 3 + 5 - 4 \\
&= 4 \text{ sq. units}
\end{aligned}$$

11. $y = x^2$



$$y = x$$

$$\Rightarrow x = 0, y = 0$$

$$x = 1, y = 1$$

$$\text{Area} = \int_0^1 x dx - \int_0^1 x^2 dx$$

$$= \int_0^1 (x - x^2) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6} \text{ sq. units}$$

12. Given $L = \lim_{x \rightarrow \infty} \left(\frac{x^x}{x!} \right)^{1/x}$

Taking logarithm on both sides

$$\log L = \lim_{x \rightarrow \infty} \frac{1}{x} \left(\log \frac{x}{1} + \log \frac{x}{2} + \dots + \log \frac{x}{x} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{r=1}^x \log \frac{x}{r}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{r=1}^x \log \frac{1}{(r/x)}$$

$$= \int_0^1 \log \frac{1}{x} dx$$

$$= - \int_0^1 \log x dx$$

$$= - [x \log x + x]_0^1$$

$$= - [(1 \log 1 + 1) - (0 \log 0 - 0)] = 1$$

$$\therefore \log L = 1$$

$$\Rightarrow L = e$$

$$\Rightarrow \lim_{x \rightarrow \infty} \left(\frac{x^x}{x!} \right)^{1/x} = e$$

13. Given, $\lim_{x \rightarrow \infty} \left[\frac{1}{x} + \frac{x^2}{(x+1)^3} + \frac{x^2}{(x+2)^3} + \dots + \frac{1}{8x} \right]$

$$= \lim_{x \rightarrow \infty} \sum_{r=0}^x \frac{x^2}{(x+r)^3}$$

$$= \lim_{x \rightarrow \infty} \sum_{r=0}^x \frac{1/x}{(1+r/x)^2}$$

$$= \int_0^1 \frac{dy}{(1+y)^3}, \text{ replace } \frac{r}{x} \text{ by } y \text{ and } \frac{1}{x} \text{ by } dy$$

$$= \left[\frac{-1}{2(1+y)^2} \right]_0^1$$

$$= \left[\frac{-1}{2(1+1^2)} - \frac{-1}{2(1+0^2)} \right]$$

$$= \left[\frac{-1}{2(2)} - \frac{-1}{2(1)} \right]$$

$$= \left[\frac{-1}{4} - \frac{-1}{2} \right] = \frac{1}{4}$$

14. We have, $x^2 = y$ and $y = x + 2$

$$\Rightarrow x^2 = x + 2$$

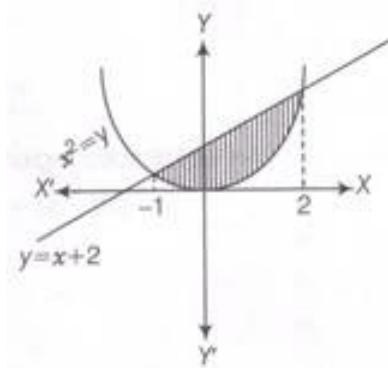
$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow x^2 - 2x + x - 2 = 0$$

$$\Rightarrow x(x-2) + 1(x-2) = 0$$

$$\Rightarrow (x+1)(x-2) = 0$$

$$\Rightarrow x = -1, 2$$



$$\therefore \text{Required area of shaded region, } = \int_{-1}^2 (x + 2 - x^2) dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2$$

$$= \left(8 - 3 - \frac{1}{2} \right) = \frac{9}{2}$$

15. Given circles are $x^2 + y^2 = 4$...(i)

$$(x - 2)^2 + y^2 = 4$$
...(ii)

Eq. (i) is a circle with centre origin and

Radius = 2.

Eq. (ii) is a circle with centre C (2, 0) and

Radius = 2.

On solving Eqs. (i) and (ii), we get

$$(x - 2)^2 + y^2 = x^2 + y^2$$

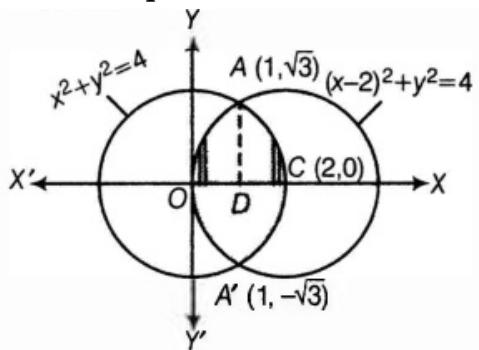
$$\Rightarrow x^2 - 4x + 4 + y^2 = x^2 + y^2$$

$$\Rightarrow x = 1$$

On putting $x = 1$ in Eq. (i), we get

$$y = \pm\sqrt{3}$$

Thus, the points of intersection of the given circles are A (1, $\sqrt{3}$) and A' (1, $-\sqrt{3}$).



Clearly, required area = Area of the enclosed region OACA'O between circles

= 2 [Area of the region ODCAO]

= 2 [Area of the region ODAO + Area of the region DCAD]

$$= 2 \left[\int_0^1 y_2 dx + \int_1^2 y_1 dx \right]$$

$$= 2 \left[\int_0^1 \sqrt{4 - (x-2)^2} dx + \int_1^2 \sqrt{4 - x^2} dx \right]$$

$$= 2 \left[\frac{1}{2}(x-2)\sqrt{4 - (x-2)^2} + \frac{1}{2} \times 4 \sin^{-1} \left(\frac{x-2}{2} \right) \right]_0^1$$

$$+ 2 \left[\frac{1}{2}x\sqrt{4 - x^2} + \frac{1}{2} \times 4 \sin^{-1} \frac{x}{2} \right]_1^2$$

$$= \left[(x-2)\sqrt{4 - (x-2)^2} + 4 \sin^{-1} \left(\frac{x-2}{2} \right) \right]_0^1 + \left[x\sqrt{4 - x^2} + 4 \sin^{-1} \frac{x}{2} \right]_1^2$$

$$= \left[\left\{ -\sqrt{3} + 4 \sin^{-1} \left(\frac{-1}{2} \right) \right\} - 0 - 4 \sin^{-1}(-1) \right] + \left[0 + 4 \sin^{-1} 1 - \sqrt{3} - 4 \sin^{-1} \frac{1}{2} \right]$$

$$= \left[\left(-\sqrt{3} - 4 \times \frac{\pi}{6} \right) + 4 \times \frac{\pi}{2} \right] + \left[4 \times \frac{\pi}{2} - \sqrt{3} - 4 \times \frac{\pi}{6} \right]$$

$$= \left(-\sqrt{3} - \frac{2\pi}{3} + 2\pi \right) + \left(2\pi - \sqrt{3} - \frac{2\pi}{3} \right)$$

$$= \frac{8\pi}{3} - 2\sqrt{3} \text{ sq units.}$$