1. Let f (x + y) = f(x) + f(y) $\forall$ x, y $\in$ **R**. Suppose that f (6) = 5 and f ' (0) = 1, then f ' (6) is equal to

    a. 1

    b. 30

    c. None of these

    d. 25

2. Derivative of log|x| w.r.t. |x| is

    a. None of these

    b. $\frac{1}{x}$

    c. $\pm\frac{1}{x}$

    d. $\frac{1}{|x|}$

3. The function f (x) = 1 + |sin x| is

    a. differentiable everywhere

    b. continuous everywhere

    c. differentiable nowhere

    d. continuous nowhere

4. $\underset{x\to 0}{Lt}\ \frac{1-\cos x}{x\sin x}$ is equal to

    a. 1

    b. 2

    c. 0

    d. $\frac{1}{2}$

5. $\underset{x\to\pi/4}{Lt}\ \frac{\cos x-\sin x}{x-\frac{\pi}{4}}$ is equal to

    a. $-\frac{2}{\sqrt{2}}$

b. -1

c. $-\dfrac{1}{\sqrt{2}}$

d. $\dfrac{2}{\sqrt{2}}$

6. The value of c in Mean value theorem for the function f(x) = x(x - 2), x $\in$[1, 2] is

    _____.

7. The set of points where the function f given by f(x) = |2x - 1| sin x is differentiable is

    _____.

8. Differential coefficient of sec (tan^{-1}x) w.r.t. x is _____.

9. Discuss the continuity of the function $f(x) = \sin x . \cos x$.

10. Determine the value of 'k' for which the following function is continuous at x = 3 : f(x)

    $= \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k & , x = 3 \end{cases}$.

11. Determine the value of the constant 'k' so that the function f(x) = $\begin{cases} \dfrac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ is

    continuous at x= 0.

12. Find $\dfrac{dy}{dx}$, $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), 0 < x < 1$

13. Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

14. Determine if f defined by $f(x) = \begin{cases} x^2 \sin\dfrac{1}{x}, & if\ x \neq 0 \\ 0,\ if\ x = 0 \end{cases}$ is a continuous function.

15. Find the value of k so that the following function is continuous at x = 2.

    f(x) = $\begin{cases} \dfrac{x^3 + x^2 - 16x + 20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{cases}$

16. If x^y + y^x = a^b, then find $\dfrac{dy}{dx}$.

17. If e^y(x + 1) = 1, then show that $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$.

18. Find $\dfrac{dy}{dx}$ if $y^x + x^y + x^x = a^b$.

Solution

1. a. 1

   Explanation: $f'(6) = \lim\limits_{h \to 0} \dfrac{f(6+h) - f(6)}{h} = \lim\limits_{h \to 0} \dfrac{f(6+h) - f(6+0)}{h}$

   $= \lim\limits_{h \to 0} \dfrac{f(6) + f(h) - \{f(6) + f(0)\}}{h}$

   $= \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h} = f'(0) = 1$

2. d. $\dfrac{1}{|x|}$

   Explanation: $\dfrac{d}{d|x|}(\log|x|) = \dfrac{1}{|x|}$

3. b. continuous everywhere

   Explanation: f(x) = 1 + |sinx| is not derivable at those x for which x for which sinx = 0, however, 1 + |sinx| is continuous everywhere (being the sum of two continuous functions)

4. d. $\dfrac{1}{2}$

   Explanation: $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x \sin x}$

   $= \lim\limits_{x \to 0} \dfrac{1 - \cos^2 x}{x \sin x (1 + \cos x)} \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \dfrac{1}{1 + \cos x} = 1 \cdot \dfrac{1}{1+1} = \dfrac{1}{2}$

5. a. $-\dfrac{2}{\sqrt{2}}$

   Explanation: $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\cos x - \sin x}{x - \frac{\pi}{4}}$

   $= \lim\limits_{x \to \frac{\pi}{4}} \dfrac{-\sin x - \cos x}{1} = -\sin\dfrac{\pi}{4} - \cos\dfrac{\pi}{4} = -\dfrac{2}{\sqrt{2}} = -\sqrt{2}$

6. $\dfrac{3}{2}$

7. R - $\left\{\dfrac{1}{2}\right\}$

8. $\dfrac{x}{\sqrt{1+x^2}}$

9. Since sin x and cos x are continuous functions and product of two continuous function is a continuous function, therefore $f(x) = \sin x . \cos x$ is a continuous function.

10. Given, f(x) = $\begin{cases} \frac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ x, & x = 3 \end{cases}$

We shall use definition of continuity to find the value of k.

If f(x) is continuous at x = 3,

Then, we have $\lim_{x \to 3} f(x) = f(3)$

$\Rightarrow \lim_{x \to 3} \frac{(x+3)^2 - 36}{x-3} = k$

$\Rightarrow \lim_{x \to 3} \frac{(x+3)^2 - 6^2}{x-3} = k$

$\Rightarrow \lim_{x \to 3} \frac{(x+3-6)(x+3+6)}{x-3} = k$ [∵ a² - b² = (a - b)(a + b)]

$\Rightarrow \lim_{x \to 3} \frac{(x-3)(x+9)}{(x-3)} = k$

$\Rightarrow \lim_{x \to 3} (x + 9) = k$

$\Rightarrow$ 3 + 9 = k $\Rightarrow$ k = 12

11. Let f(x) = $\begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ be continuous at x = 0

Then, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0)$

$\Rightarrow \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(0 - h) = f(0)$

$\Rightarrow 3 = \lim_{h \to 0} \frac{k(-h)}{|-h|} = 3$

$\Rightarrow \lim_{h \to 0} \left(\frac{-kh}{h} \right) = 3$

$\lim_{h \to 0} (-k) = 3$

∴ k = - 3

12. Given: $y = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right), 0 < x < 1$

Putting $x = \tan \theta$

$y = \cos^{-1} \left(\frac{1-\tan^2 \theta}{1+\tan^2 \theta} \right)$

$= \cos^{-1} (\cos 2\theta) = 2\theta = 2\tan^{-1} x$

∴ $\frac{dy}{dx} = 2 \cdot \frac{1}{1+x^2} = \frac{2}{1+x^2}$

13. Let $f(x) = x^2$ and $g(x) = \cos x$, then

$(gof)(x) = g[f(x)] = g(x^2) = \cos x^2$

Now f and g being continuous it follows that their composite (gof) is continuous.

Hence $\cos x^2$ is continuous function.

14. Here, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$ x a finite quantity = 0

$\left[\because \sin\frac{1}{x} \text{ lies between - 1 and 1}\right]$

Also f(0) = 0

Since, $\lim\limits_{x\to 0} f(x) = f(0)$ therefore, the function f is continuous at x = 0.

Also,when $x \neq 0$,then f(x) is the product of two continuous functions and hence Continuous.Hence,f(x) is continuous everywhere.

15. According to the question, f(x) = $\left\{\begin{array}{ll} \frac{x^3+x^2-16x+20}{(x-2)^2}, & x \neq 2 \\ k, & x = 2 \end{array}\right\}$ is continuous at $x = 2$.

Now, we have f(2) = k

$\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} \frac{x^3+x^2-16x+20}{(x-2)^2}$

$= \lim\limits_{x\to 2} \frac{(x-2)(x^2+3x-10)}{(x-2)^2}$

$= \lim\limits_{x\to 2} \frac{(x-2)(x+5)(x-2)}{(x-2)^2}$

$= \lim\limits_{x\to 2}(x+5)$ = 2+ 5 = 7

f(x) is continuous at x = 2.

$\therefore \lim\limits_{x\to 2} f(x)$ = f(2) $\Rightarrow$ 7 = k $\Rightarrow$ k = 7

16. We have, $x^y + y^x = a^b$.........(i)

Let x^y = v and y^x = u......(ii)

Therefore,on putting these values in Eq. (i), we get,

v + u = a^b

Therefore,on differentiating both sides w.r.t. x, we get,

$\frac{dv}{dx} + \frac{du}{dx} = 0$........(iii)

Now consider, x^y = v [from Eq.(ii)]

Therefore,on taking log both sides, we get,

log x^y = logv

$\Rightarrow$ y log x = log v

Therefore,on differentiating both sides w.r.t. x, we get,

$y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx} = \frac{1}{v}\frac{dv}{dx}$

$\Rightarrow v\left(\frac{y}{x} + \log x \cdot \frac{dy}{dx}\right) = \frac{dv}{dx}$

$\Rightarrow \quad \frac{dv}{dx} = x^y\left(\frac{y}{x} + \log x \frac{dy}{dx}\right)$.........(iv) [From Eq.(ii)]

Also, $y^x = u$ [From Eq(ii)]

Therefore, on taking log both sides, we get,

$\log y^x = \log u \Rightarrow x \log y = \log u$

Therefore, on differentiating both sides w.r.t. 'x', we get,

$x \cdot \frac{1}{y} \frac{dy}{dx} + 1 \cdot \log y = \frac{1}{u} \frac{du}{dx}$

$\Rightarrow \quad \frac{x}{y} \frac{dy}{dx} + \log y = \frac{1}{u} \frac{du}{dx}$

$\Rightarrow \quad u \left[\frac{x}{y} \frac{dy}{dx} + \log y \right] = \frac{du}{dx}$

$\Rightarrow \quad y^x \left[\frac{x}{y} \frac{dy}{dx} + \log y \right] = \frac{du}{dx}$ ....(v) [From Eq(ii)]

Therefore, on substituting the values of $\frac{dv}{dx}$ and $\frac{du}{dx}$ from Eqs. (iv) and (v) respectively in Eq. (iii), we get

$x^y \left(\frac{y}{x} + \log x \cdot \frac{dy}{dx} \right) + y^x \left(\frac{x}{y} \frac{dy}{dx} + \log y \right) = 0$

$\Rightarrow x^y \frac{y}{x} + x^y \log x \cdot \frac{dy}{dx} + y^x \cdot \frac{x}{y} \frac{dy}{dx} + y^x \log y = 0$

$\Rightarrow x^y \log x \cdot \frac{dy}{dx} + y^x \frac{x}{y} \cdot \frac{dy}{dx} = -x^y \frac{y}{x} - y^x \log y$

$\Rightarrow \quad \frac{dy}{dx} \left[x^y \log x + y^x \cdot \frac{x}{y} \right] = -x^y \cdot \frac{y}{x} - y^x \log y$

$\therefore \quad \frac{dy}{dx} = \frac{-x^{y-1} \cdot y - y^x \log y}{x^y \log x + y^{x-1} \cdot x}$

17. According to the question, $e^y (x + 1) = 1$

Taking log both sides,

$\Rightarrow \log [e^y (x + 1)] = \log 1$

$\Rightarrow \log e^y + \log (x + 1) = \log 1$

$\Rightarrow y + \log (x + 1) = \log 1$ [∵ $\log e^y = y$]

differentiating both sides w.r.t. x,

$\Rightarrow \frac{dy}{dx} + \frac{1}{x+1} = 0$ ....(i)

Differentiating both sides w.r.t. 'x',

$\Rightarrow \frac{d^2 y}{dx^2} - \frac{1}{(x+1)^2} = 0$

$\Rightarrow \quad \frac{d^2 y}{dx^2} - \left(-\frac{dy}{dx} \right)^2 = 0$ [From Equation(i)]

$\Rightarrow \quad \frac{d^2 y}{dx^2} - \left(\frac{dy}{dx} \right)^2 = 0$

$\Rightarrow \quad \frac{d^2 y}{dx^2} = \left(\frac{dy}{dx} \right)^2$

18. Let $u = y^x, v = x^y, w = x^x$

$u + v + w = a^b$

Therefore $\frac{du}{dx} + \frac{dw}{dx} + \frac{dv}{dx} = 0$ ....(1)

$u = y^x$

Taking log both side

$\log u = \log y^x$

$\log u = x.\log y$

Differentiate both side w.r.t. to x

$\frac{1}{u}.\frac{du}{dx} = x.\frac{1}{y}.\frac{dy}{dx} + \log y.1$

$\frac{du}{dx} = u\left[\frac{x}{y}.\frac{dy}{dx} + \log y\right]$

$\frac{du}{dx} = y^x\left[\frac{x}{y}.\frac{dy}{dx} + \log y\right]$ .... (2)

$v = x^y$

Taking log both side

$\log v = \log x^y$

$\log v = y.\log x$

$\frac{1}{v}.\frac{dv}{dx} = y.\frac{1}{x} + \log x.\frac{dy}{dx}$

$\frac{dv}{dx} = v\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right]$

$\frac{dv}{dx} = x^y\left[\frac{y}{x} + \log x.\frac{dy}{dx}\right]$ .... (3)

$w = x^x$

Taking log both side

$\log w = \log x^x$

$\log w = x \log x$

$\frac{1}{w}.\frac{dw}{dx} = x.\frac{1}{x} + \log x.1$

$\frac{1}{w}.\frac{dw}{dx} = 1 + \log x$

$\frac{dw}{dx} = w(1 + \log x)$

$\frac{dw}{dx} = x^x(1 + \log x)$ .... (4)

$\frac{dy}{dx} = \frac{-x^x(1+\log x) - y.x^{y-1} - y^x \log y}{x.y^{x-1} + x^y \log x.}$ (by putting 2,3 and 4 in 1)